Vector Sampling Expansion

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***Abstract*—The vector sampling expansion (VSE) is an extension of Papoulis' generalized sampling expansion (GSE) to the vector case. In VSE, N bandlimited signals, all with the same bandwidth B, are passed through a multi-input–multi-output (MIMO) linear time invariant system that generates M ($M \geq N$) output signals. The goal is to reconstruct the input signals from the samples of the output signals at a total sampling rate of N times Nyquist rate, where the Nyquist rate is B/π samples per second. We find necessary and sufficient conditions for this reconstruction. A surprising necessary condition for the case where all output signals are uniformly sampled at the same rate (N/M times the Nyquist rate) is that the expansion factor M/N must be an integer. This condition is no longer necessary when each output signal is sampled at a different rate or sampled nonuniformly. This work also includes a noise sensitivity analysis of VSE systems. We define the noise amplification factor, which allows a quantitative comparison between VSE systems, and determine the optimal VSE systems.**

***Index Terms*—Generalized sampling expansion, nonuniform sampling, quantization, sensitivity, signal reconstruction, signal sampling, vector sampling.**

I. Introduction

In HIS famous generalized sampling expansion (GSE) [1], [2], Papoulis has shown that a bandlimited signal $f(t)$ of finite energy, passing through M linear time-invariant (LTI) systems and generating responses $g_k(t)$, $k = 1, \cdots, M$, can be uniquely reconstructed, under some conditions on the M filters, from the samples of the output signals $g_k(nT)$, sampled at $1/M$ the Nyquist rate. Such a sampling scheme might be useful when the original signal is not directly accessible, but some processed versions of it exist and may be used for reconstruction. More recently [3], [4], the GSE has been extended to multidimensional signals in which the signal f depends on several variables, i.e., $f(\boldsymbol{x}) = f(x_1, \cdots, x_K)$. This work provides another *vector* extension to the GSE: the vector sampling expansion (VSE).

We consider N bandlimited signals, or a signal vector $\boldsymbol{f}(t)^T = [f_1(t), \cdots, f_N(t)]$, all having the same bandwidth B that pass through a multi-input–multi-output (MIMO) LTI system, as in the left-hand side of Fig. 1, to yield M output signals $\boldsymbol{g}(t)^T = [g_1(t), \cdots, g_M(t)]$, where $M \geq N$. The transfer function of the MIMO system is denoted $\boldsymbol{H}(\omega)$, where $\boldsymbol{H}(\omega)$ is an $M \times N$ matrix, and therefore, we have

$$\boldsymbol{G}(\omega) = \boldsymbol{H}(\omega)\boldsymbol{F}(\omega) \tag{1}$$

where

$$\boldsymbol{F}(\omega)^T = [F_1(\omega), \cdots, F_N(\omega)]$$
$$\boldsymbol{G}(\omega)^T = [G_1(\omega), \cdots, G_M(\omega)]$$

and $F_i(\omega)$, $G_j(\omega)$ are the Fourier transforms of $f_i(t)$, $g_j(t)$, respectively.

We examine whether the N input signals can be reconstructed from samples of the M output signals at rates that preserves the total rate to be N times Nyquist rate (the rate obtained by sampling each of the input signals at the Nyquist rate). A VSE system, which is described in Fig. 1, is a system where such reconstruction is possible. We will provide in this paper the necessary conditions for such reconstructions in several cases. The conditions we provide are for signals with no known deterministic functional relationship between them since dependency between the signals, if known, can be utilized to further reduce the required sampling rate.

VSE systems appear in many practical applications. For example, in a multiaccess wireless communication environment or radar/sonar environment, we may have N transmitters, emitting N different signals, which are then received by M antennas. A question of great interest is how to sample the received signals at the total minimal rate that will enable unique reconstruction and how to attain the most noise-robust system. VSE systems also appear whenever the information is represented by a vector signal. For example, consider an RGB color image, and suppose we use, say, four color filters to acquire it. Sampling the filters output and reconstructing the RGB image is a VSE system. In this example, the problem of determining the proper arrangement of the filters on the sensor is equivalent to determining appropriate sampling scheme of the VSE system.

The paper is organized as follows. In Section II, we discuss the case of equal uniform sampling of all output channels. It turns out, somewhat surprisingly, that there is a distinction between expansion by an integer factor (i.e., M/N is an integer) and expansion by a noninteger factor. When all the output signals are uniformly sampled at the same rate (which is N/M times the Nyquist rate), we show that reconstruction of $\boldsymbol{f}(t)$ is possible, with some conditions on the MIMO system, if and only if the expansion factor M/N is an integer. In this section, we also find the reconstruction formula and discuss the stochastic signal case. The main results of Section II were also summarized in [5].

Reconstruction is also possible when M/N is not an integer. However, in this case, either the sampling rate is not equal for all output signals, or each output signal is sampled nonuniformly. Uniform sampling at different sampling rates for different output signals is discussed in Section III, whereas periodic nonuniform sampling is discussed in Section IV.

Manuscript received July 27, 1998; revised August 20, 1999. The associate editor coordinating the review of this paper and approving it for publication was Dr. Ali H. Sayed.

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Publisher Item Identifier S 1053-587X(00)02362-X.

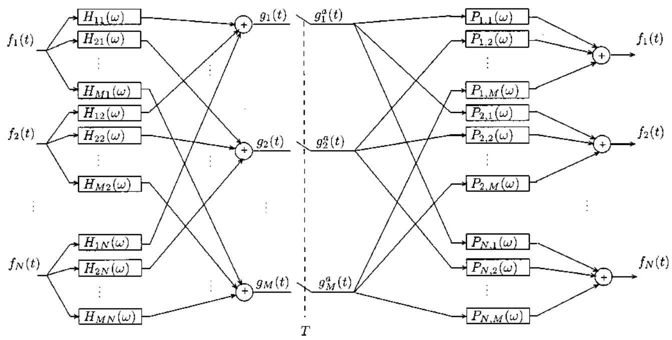

Fig. 1. Complete VSE system.

In Section V, we analyze the performance of VSE systems in the presence of white quantization noise. We find the necessary and sufficient condition for an optimal VSE system in the sense of having a minimum mean square reconstruction error.

II. Equal Uniform Sampling

In this section, we discuss the case of equal uniform sampling in all M output channels of the MIMO system. We show that a unique reconstruction is possible only when M/N is an integer. We then find reconstruction formulas for this case in time and frequency domains. Finally, we discuss the stochastic signal case.

A. Expansion by an Integer Factor

Consider the case where $M/N = m$ is an integer. When sampling at N/M the Nyquist rate, i.e., at a sampling period $T = M\pi/NB$ (B is the bandwidth), we get aliased versions of the output signals, which, at the frequency domain, are periodic with a period $c = 2B/m$. We denote by $G_k^a(\omega)$ the Fourier transform of the sampled kth output signal and observe that since it is periodic with a period c, it is sufficient to consider only one period, say $\omega \in [-B, -B+c]$. In this region, $G_k^a(\omega)$ is composed of m replicas of $G_k(\omega)$, the Fourier transform of the kth output signal, shifted in frequency by multiples of c, i.e.,

$$G_k^a(\omega) = \frac{c}{2\pi} \sum_{i=0}^{m-1} G_k(\omega + ic), \qquad \omega \in [-B, -B+c]. \quad (2)$$

Since $G_k(\omega) = \sum_{l=1}^{N} H_{kl}(\omega) F_l(\omega)$, where $H_{kl}(\omega)$ is the (k, l)th component of the MIMO system transfer matrix $\boldsymbol{H}(\omega)$, we have

$$G_k^a(\omega) = \frac{c}{2\pi} \sum_{i=0}^{m-1} \sum_{l=1}^{N} H_{kl}(\omega + ic) F_l(\omega + ic). \quad (3)$$

This is true for $k = 1, 2, \cdots, M$, and therefore, we may write, in a matrix form

$$\boldsymbol{G}^a(\omega) = \frac{c}{2\pi} \boldsymbol{T}(\omega) \boldsymbol{F}^a(\omega) \qquad \omega \in [-B, -B+c] \quad (4)$$

where $\boldsymbol{G}^a(\omega)^T = [G_1^a(\omega), G_2^a(\omega), \cdots, G_M^a(\omega)]$, $\boldsymbol{F}^a(\omega)$ is the M-dimensional vector

$$\begin{aligned}\boldsymbol{F}^a(\omega)^T = &[F_1(\omega), F_1(\omega + c), \cdots, F_1(\omega + (m-1)c) \\ &\cdots, F_N(\omega), \cdots, F_N(\omega + (m-1)c)]\end{aligned} \quad (5)$$

i.e., its lth component $F_l^a(\omega) = F_{l_1}(\omega + (l_2 - 1)c)$ where $l_1 = \lceil l/m \rceil$ and

$$l_2 = \begin{cases} (l \bmod m), & m \text{ does not divide } l \\ m, & m \text{ divides } l. \end{cases}$$

Finally, $\boldsymbol{T}(\omega)$ is an $M \times M$ matrix whose (k, l)th component is given by

$$T_{kl}(\omega) = H_{k, l_1}(\omega + (l_2 - 1) \cdot c). \quad (6)$$

We observe that (4) is a set of M equations for the $mN = M$ unknowns $F_l(\omega + ic)$, where $l = 1, \cdots, N$, and $i = 0, \cdots, (m-1)$. By solving this system of equations, we get the Fourier transform of the input signals at all frequencies $\omega \in [-B, B]$, i.e., we can reconstruct the input signals. Note that this system of equations will have a single solution if the determinant of the matrix $\boldsymbol{T}(\omega)$, which depends solely on the MIMO system, is not zero for every $\omega \in [-B, -B+c]$. Many MIMO systems satisfy this condition, but it should be checked to determine if reconstruction is possible.

One simple example that enables reconstruction is as follows. Let $\boldsymbol{H}_1$ be an $M \times N$ constant matrix of rank N. As discussed above, if this constant matrix is the MIMO transfer function, reconstruction is impossible since at any sampling time point, we get dependent samples. Suppose, however, that we stagger the signals, i.e., shift the kth output signal by $(k-1)T/M = (k-1)\pi/NB$ and then sample each output signal at sampling period T. This is equivalent to sampling at N times the Nyquist rate while multiplexing between the M output signals. The transfer function of the MIMO system in this case is $\boldsymbol{H}(\omega) = \boldsymbol{D}(\omega)\boldsymbol{H}_1$, where $D(\omega) = \operatorname{diag}\{1, e^{j\omega T/M}, \cdots, e^{j\omega(M-1)T/M}\}$. It is easy to see that in this case, the resulting $\boldsymbol{T}(\omega)$ has a full rank for all ω, and therefore, reconstruction is possible.

We next show that we can get such a solvable set of equations for all the frequency content of the input signals only when M/N is an integer, implying that this is a necessary condition for reconstruction.

Suppose M/N is not an integer but that $\tilde{m} < M/N < \tilde{m}+1$, where $\tilde{m}$ is an integer. As we sample, say, the output signal $g_k(t)$ at every $T = M\pi/NB$, we get an aliased (sampled) signal whose period in the frequency domain is still $2BN/M$. Again, we choose as the basic period the interval $[-B, -B + 2BN/M]$. This interval can be further divided to N intervals of size $2B/M$ each. We see that in the first $(M \bmod N)$ of these N intervals, the Fourier transform of the sampled signals $G_k^a(\omega)$ is composed of $\tilde{m}+1$ replicas of $G_k(\omega)$, whereas in the rest of the $N - (M \bmod N)$ intervals, there are only $\tilde{m}$ replicas. This situation is illustrated in Fig. 2 for the case $N = 2$, $M = 3$. For the frequencies where there are only $\tilde{m}$ replicas, we have M equations (an equation for each output signal) for $\tilde{m}N$ unknowns (the unknowns are the $\tilde{m}$ replicas of each of the N input signals). Because $\tilde{m}N < M$, there are more equations than needed for a solution in this interval. This means that we somehow wasted samples in this frequency interval, which will cause a shortage of samples for the other frequency intervals (recall that the total sampling rate is exactly N times the Nyquist rate). Indeed, in the frequency intervals where there are $\tilde{m}+1$

replicas, we have $(\tilde{m}+1)N$ unknowns but only M equations, and since $M < (\tilde{m}+1)N$, the set of $(\tilde{m}+1)N$ equations does not have a single solution, but there is a space of many possible solutions. Since it is assumed that no known functional dependency between the N input signals exists, there are no additional conditions to determine a unique reconstruction of the input signals. In summary, we do not have enough information to reconstruct the input signals in this case, where M/N is not integer, and all outputs are sampled at the same rate.

B. The VSE Interpolation Formula

In this section, we provide the explicit interpolation formula for the case where $M/N = m$ is an integer, and reconstruction is possible. This derivation resembles the technique used in [4] and [6].

The first step is to write an explicit expression for the input signal $f_i(t)$ in terms of its aliased components $F_i(\omega), \cdots, F_i(\omega + (m-1)c)$, which, as described above, can be reconstructed for $\omega \in [-B, -B+c]$. Then, we use the inverse Fourier transform formula and a simple change of variables to write

$$\begin{aligned} f_i(t) &= \frac{1}{2\pi}\int_{-B}^{B} F_i(\omega)e^{j\omega t}\,d\omega \\ &= \sum_{k=0}^{m-1}\frac{1}{2\pi}\int_{-B+kc}^{-B+kc+c} F_i(\omega)e^{j(\omega)t}\,d\omega \\ &= \sum_{k=0}^{m-1}\frac{1}{2\pi}\int_{-B}^{-B+c} F_i(\omega+kc)e^{j(\omega+kc)t}\,d\omega. \end{aligned} \tag{7}$$

This relation can be expressed in terms of $\boldsymbol{F}^a(\omega)$, which is the vector defined in (5), i.e.,

$$f_i(t) = \frac{1}{2\pi}\int_{-B}^{-B+c} \boldsymbol{E}_i(t)^T\boldsymbol{F}^a(\omega)e^{j\omega t}\,d\omega \tag{8}$$

where $\boldsymbol{E}_i(t)^T$ is an M-dimensional vector whose kth component is nonzero and equals $e^{j(k-1-m(i-1))ct}$ only in the region $(i-1)m < k \le im$ [i.e., at this region, it takes the values $1, e^{jct}, \cdots, e^{j(m-1)ct}$], and it is zero elsewhere. Note that since $cT = 2\pi$, $\boldsymbol{E}_i(t) = \boldsymbol{E}_i(t-nT)$ for any integer n, i.e., it is periodic with period T, where $T = M\pi/NB$.

We now define a set of M-dimensional vectors $\boldsymbol{Y}_i(\omega, t)^T = [Y_{i,1}(\omega, t), \cdots, Y_{i,M}(\omega, t)]$ as the solutions of

$$\boldsymbol{T}(\omega)^T\boldsymbol{Y}_i(\omega, t) = \boldsymbol{E}_i(t) \qquad \omega \in [-B, -B+c] \tag{9}$$

where $\boldsymbol{T}(\omega)$, which is defined in (6), is assumed to be invertible at each $\omega \in [-B, -B+c]$ to assure reconstruction. Note that since $\boldsymbol{E}_i(t)$ is periodic, $\boldsymbol{Y}_i(\omega, t)$ is also periodic in t with period T.

Equation (10), shown at the top of the next page, is a more detailed representation of (9). The $M \times M$ matrix $\boldsymbol{T}(\omega)^T$ is composed of N matrices of size $m \times M$. The ith matrix, which corresponds to the m equations for which we have nonzero values in $\boldsymbol{E}_i(t)$, is similar to Papoulis' original system of equations, only that Papoulis' matrix is of size $M \times M$. The reason is that in our case, each output channel is sampled N times faster than

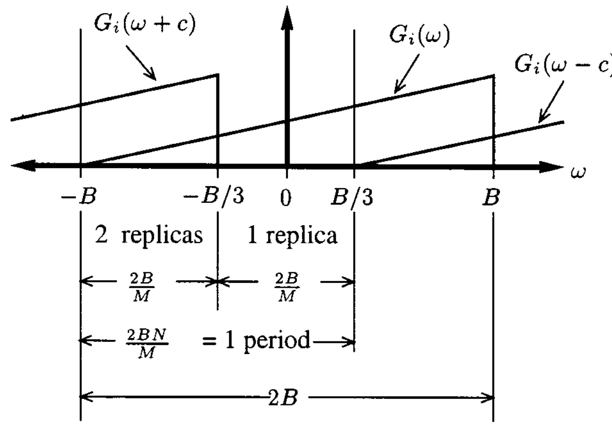

Fig. 2. Components of the ith output channel in the frequency domain when $N = 2$ and $M = 3$.

in the GSE case, and therefore, the number of equations is reduced by a factor of N, i.e., there are only $m = M/N$ equations. The missing $(M-m)$ equations are obtained by forcing the output of the reconstructed ith channel to be independent on the other $(N-1)$ input channels. These equations correspond to the $(M-m)$ zero components of $\boldsymbol{E}_i(t)$. The GSE is, of course, a particular case of the VSE when $N = 1$.

Since $\boldsymbol{E}_i(t)^T = \boldsymbol{Y}_i(\omega, t)^T\boldsymbol{T}(\omega)$, and using the relation (4), (8) becomes

$$\begin{aligned} f_i(t) &= \frac{1}{2\pi}\int_{-B}^{-B+c} \boldsymbol{Y}_i(\omega, t)^T\boldsymbol{T}(\omega)\boldsymbol{F}^a(\omega)e^{j\omega t}\,d\omega \\ &= \frac{1}{c}\int_{-B}^{-B+c} \boldsymbol{Y}_i(\omega, t)^T\boldsymbol{G}^a(\omega)e^{j\omega t}\,d\omega \end{aligned} \tag{11}$$

which is the interpolation formula expressing the input signals in terms of the Fourier transform of the aliased sampled signals.

To get a formula in the time domain, we define the signals

$$y_{i,k}(t) = \frac{1}{c}\int_{-B}^{-B+c} Y_{i,k}(\omega, t)e^{j\omega t}\,d\omega. \tag{12}$$

Note that $y_{i,k}(t)$ is not periodic, despite the fact that $Y_{i,k}(\omega, t)$ is periodic in t. Now, the Fourier transform of the sampled signal is given by

$$G_k^a(\omega) = \sum_{n=-\infty}^{\infty} g_k(nT)e^{-j\omega nT}. \tag{13}$$

Thus, substituting (12) and (13) in (11) and using the fact that $\boldsymbol{Y}_i(\omega, t)$ is periodic in t with period T, we get the interpolation formula in the time domain

$$f_i(t) = \sum_{k=1}^{M}\left[\sum_{n=-\infty}^{\infty} g_k(nT)y_{i,k}(t-nT)\right]. \tag{14}$$

This equation describes a sum of M convolutions of the M sampled sequences with the signals $y_{i,k}(t)$, which are calculated by (12) from the vectors $\boldsymbol{Y}_i(\omega, t)$ that depend solely on the MIMO system via the relation (9).

We can write this result in a matrix form as

$$\boldsymbol{f}(t) = \boldsymbol{y}(t) * \boldsymbol{g}^a(t) \tag{15}$$

$$\begin{bmatrix} H_{11}(\omega) & H_{21}(\omega) & \cdots & H_{M1}(\omega) \\ H_{11}(\omega+c) & H_{21}(\omega+c) & \cdots & H_{M1}(\omega+c) \\ H_{11}(\omega+2c) & H_{21}(\omega+2c) & \cdots & H_{M1}(\omega+2c) \\ \vdots & \vdots & & \vdots \\ H_{11}(\omega+(m-1)c) & H_{21}(\omega+(m-1)c) & \cdots & H_{M1}(\omega+(m-1)c) \\ H_{12}(\omega) & H_{22}(\omega) & \cdots & H_{M2}(\omega) \\ \vdots & \vdots & & \vdots \\ H_{12}(\omega+(m-1)c) & H_{22}(\omega+(m-1)c) & \cdots & H_{M2}(\omega+(m-1)c) \\ \vdots & \vdots & & \vdots \\ H_{1N}(\omega) & H_{2N}(\omega) & \cdots & H_{MN}(\omega) \\ \vdots & \vdots & & \vdots \\ H_{1N}(\omega+(m-1)c) & H_{2N}(\omega+(m-1)c) & \cdots & H_{MN}(\omega+(m-1)c) \end{bmatrix} \begin{bmatrix} Y_{i,1}(\omega, t) \\ Y_{i,2}(\omega, t) \\ \vdots \\ Y_{i,M}(\omega, t) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ e^{jct} e^{j2ct} \\ \vdots \\ e^{j(m-1)ct} \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{10}$$

where $\boldsymbol{g}^a(t)^T = [g_1^a(t), \cdots, g_M^a(t)]$, $g_k^a(t) = \sum_{n=-\infty}^{\infty} g_k(nT)\delta(t-nT)$, $\boldsymbol{y}(t)$ is the matrix of signals whose (i, k)th component is $y_{i,k}(t)$. The symbol $*$ in (15) means that convolutions are performed instead of multiplications in the matrix-vector multiplication.

C. Frequency Domain Solution

Brown [7] showed that the Fourier transform of $y_k(t)$, i.e., the frequency response of the filter that operate on the k output in a GSE reconstruction system, can be found directly, without calculating $Y_k(\omega, t)$. We derive a similar formula for the VSE.

We denote the Fourier transform of $y_{i,k}(t)$ by $P_{i,k}(\omega)$. We also define the matrix $\boldsymbol{P}(\omega)$, where the $(k, (i-1)M/N+l+1)$th component of $\boldsymbol{P}(\omega)$ is $P_{i,k}(\omega+lc)$. This is a slice of width c in ω of the Fourier transform of $y_{i,k}(t)$ of (12), which is shifted left in ω by lc. A closer look shows that we have

$$\boldsymbol{F}^a(\omega) = \boldsymbol{P}(\omega)^T \boldsymbol{G}^a(\omega) \qquad \omega \in [-B, -B+c]. \tag{16}$$

Using (4), we immediately see that

$$\boldsymbol{P}(\omega)^T = \frac{2\pi}{c} \boldsymbol{T}(\omega)^{-1} \tag{17}$$

which is a frequency domain reconstruction formula similar to Brown's formula. Note that the Fourier transforms of the reconstruction filters can be found directly from the columns of the inverse of $\boldsymbol{T}(\omega)^T$. We use this representation later on in Section V for noise sensitivity analysis.

D. The Discrete Signal Case

The discrete equivalent to Papoulis' GSE is the "alias-free QMF bank" or the "perfect reconstruction QMF bank," which have been discussed by many [8]. This equivalence has been utilized by Vaidyanathan and Liu [9], who developed, for example, sampling theorems for nonuniform decimation of discrete time sequences. This resembles using the GSE to prove Yen's periodic nonuniform sampling [16]. A similar perfect reconstruction (PR) condition for the discrete VSE case has been derived in [10]. The PR condition mentioned in [8, eq. (64)] and [9, eq. (8)] is a particular case of a discrete VSE for $N = 1$.

It is also shown in [10] that the PR condition of the discrete VSE is a special case of the continuous frequency domain solution given by equation (17), in which the filters of the MIMO system are periodic in ω with periods of $2B$. Thus, the continuous case is more general.

Purely speaking, in the discrete time world decimation is only by an integer factor, and therefore, in the accurate discrete equivalent to the VSE, M/N is integer. Thus, the problem of noninteger M/N and other issues discussed in the paper are relevant only for the continuous-time case.

E. Stochastic Signal Case

We now discuss the interpolation formula for the case where the inputs are N bandlimited wide-sense stationary (WSS) processes $x_i(t)$, $i = 1, \cdots, N$. We can reconstruct the input process $x_i(t)$ by using, e.g., (14). The reconstructed input $x_i^r(t)$ will be equal in the mean square sense to $x_i(t)$, i.e., $E\{|x_i(t) - x_i^r(t)|^2\} = 0$. To prove that, we derive the interpolation formula (14) using another technique that enables the analysis of the stochastic signal case. This technique follows Papoulis [1],[2], [11].

We begin by looking at (9) or its equivalent (10), corresponding to the reconstruction of the ith input signal. The matrix equation (10) can be divided to N groups of m equations. The uth equation of the qth group in case $q \neq i$ is

$$\sum_{k=1}^{M} H_{kq}(\omega+uc) Y_{i,k}(\omega, t) = 0 \tag{18}$$

where $u = 0, \cdots, (m-1)$, $q = 1, \cdots, N$ $q \neq i$. This equation thus corresponds to $N-1$ groups. As for the ith group, i.e., the case where $q = i$, the uth equation is

$$\sum_{k=1}^{M} H_{ki}(\omega+uc) Y_{i,k}(\omega, t) = e^{juct} \tag{19}$$

where again, $u = 0, \cdots, (m-1)$. We first discuss (19). The signal $Y_{i,k}(\omega, \tau)e^{j\omega\tau}$, in the interval $[-B, -B+c]$, considered periodic in ω with period c, can be expanded into a Fourier series. Using (12) and the periodicity of $Y_{i,k}(\omega, \tau)$ in τ, we see that the coefficients of the expansion are the $y_{i,k}(\tau - nT)$'s. We can therefore write

$$Y_{i,k}(\omega, \tau)e^{j\omega\tau} = \sum_{n=-\infty}^{\infty} y_{i,k}(\tau - nT)e^{j\omega nT}. \tag{20}$$

We now replace t by τ in (19). Multiplying the m equations corresponding to the choice $q = i$ by $e^{j\omega\tau}$ and using (20), we have new m equations

$$\sum_{k=1}^{M} H_{ki}(\omega + uc) \sum_{n=-\infty}^{\infty} y_{i,k}(\tau - nT)e^{j\omega nT} = e^{j(\omega+uc)\tau}. \tag{21}$$

This is true for $u = 0, \cdots, (m-1)$ and for every $\omega \in [-B, -B+c]$. Using the identity $e^{j\omega nT} = e^{j(\omega+uc)nT}$ and substituting $(\omega + uc)$ for ω, for every u, we conclude that these m equations may be represented by a single equation that holds in the entire interval $[-B, B]$. Thus, we have

$$\sum_{k=1}^{M} H_{ki}(\omega) \sum_{n=-\infty}^{\infty} y_{i,k}(\tau - nT)e^{j\omega nT} = e^{j\omega\tau} \tag{22}$$

for every $\omega \in [-B, B]$. The right-hand side of this equation is the frequency response of an LTI system corresponding to a time shift τ. The left-hand side is a sum of terms of the form $H_{ki}(\omega)y_{i,k}(\tau - nT)e^{j\omega nT}$. Each term is the frequency response of an LTI system, whose response to an input $f_i(t)$ will be $g_{ki}(t + nT)y_{i,k}(\tau - nT)$, where

$$g_{ki}(t) = h_{ki}(t) * f_i(t). \tag{23}$$

Thus, with an input $f_i(t)$, we get from (22) in the time domain

$$f_i(t+\tau) = \sum_{k=1}^{M}\left[\sum_{n=-\infty}^{\infty} g_{ki}(t+nT)y_{i,k}(\tau - nT)\right]. \tag{24}$$

For the cases where $q \neq i$, we get that

$$0 = \sum_{k=1}^{M}\left[\sum_{n=-\infty}^{\infty} g_{kq}(t+nT)y_{i,k}(\tau - nT)\right], \qquad q \neq i. \tag{25}$$

From the definition of the MIMO system (1), we recall that

$$g_k(t) = \sum_{q=1}^{N} h_{kq}(t) * f_q(t) = \sum_{q=1}^{N} g_{kq}(t). \tag{26}$$

Adding (24) to the $N-1$ equations of (25) and using (26), we get

$$f_i(t+\tau) = \sum_{k=1}^{M}\left[\sum_{n=-\infty}^{\infty} g_k(t+nT)y_{i,k}(\tau - nT)\right]. \tag{27}$$

Choosing t to be zero and exchanging τ and t leads to the interpolation formula (14).

For the case where the inputs are N bandlimited wide-sense stationary (WSS) processes $x_i(t)$, $i = 1, \cdots, N$, (22) still describes two LTI systems. We recall that two linear systems that have the same frequency response and are fed by the same bandlimited WSS input generate two outputs that are equal in the mean square sense ([11, eq. (11–126)]). Specifically, if a WSS processes $x_i(t)$ is the input to the two systems described by (22), the two outputs will be equal in the mean square sense. Therefore

$$x_i(t+\tau) \overset{ms}{=} \sum_{k=1}^{M}\left[\sum_{n=-\infty}^{\infty} g_{ki}(t+nT)y_{i,k}(\tau - nT)\right] \tag{28}$$

where $g_{ki}(t)$ (which is now a WSS stochastic process) is the output of the $h_{ki}(t)$ LTI filter fed by $x_i(t)$, and the equality is in the mean square sense. Using this reasoning, and following the derivation of (27), we also get

$$x_i(t+\tau) \overset{ms}{=} \sum_{k=1}^{M}\left[\sum_{n=-\infty}^{\infty} g_k(t+nT)y_{i,k}(\tau - nT)\right] \tag{29}$$

where the right-hand side is $x_i^r(t+\tau)$, which is the reconstruction of $x_i(t+\tau)$. Choosing $t = 0$ and exchanging τ with t concludes the proof for the stochastic signal case.

III. Nonequal Uniform Sampling

In this section, we consider the case where we sample the kth output of the MIMO system every $T_k = m_k T$, where m_k is a rational number $m_k = a_k/b_k$, and where a_k and b_k are relatively prime, i.e., $\mathrm{GCD}(a_k, b_k) = 1$, where GCD is the greatest common divisor, and where $T = \pi/B$. This is depicted in Fig. 3. The total sampling rate is N times the Nyquist rate, i.e.,

$$N = \sum_{k=1}^{M} \frac{1}{m_k}. \tag{30}$$

We define m as the least common multiplier (LCM) of the a_k's and, as usual, denote $c = 2B/m$.

The period of $G_k^a(\omega)$ in ω is $c_k = 2B/m_k$. The choice of m as the LCM of the a_k's assures that c_k is an integer multiple of c. This means that in the kth output channel, we have $s_k = c_k/c = m/m_k$ intervals of size c in one period. For each interval, we can write one equation in the frequency domain that can be transferred to $\omega \in [-B, -B+c]$ by change of variables. Therefore, the total number of these intervals is the number of equations available. As can be seen below, this number is mN:

$$\sum_{k=1}^{M} s_k = \sum_{k=1}^{M} \frac{m}{m_k} = m\sum_{k=1}^{M}\frac{1}{m_k} = mN.$$

We have m such intervals in the whole bandwidth $[-B, B]$ for all of the N input signals. The Fourier transforms of these mN intervals are the unknowns. Therefore, we have a system of mN equations and mN unknowns. If the matrix $\boldsymbol{T}(\omega)$ representing this system is invertible in $\omega \in [-B, -B+c]$, we can uniquely determine the N input signals. Fig. 4 demonstrates the intervals (unknowns) and the equations in a specific example of $N = 2$, $M = 4$, $m_1 = 4/3$, $m_2 = 3/2$, $m_3 = 3$, and $m_4 = 4$.

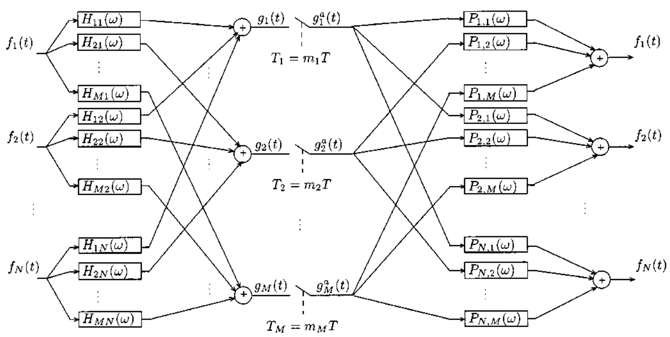

Fig. 3. VSE system using nonequal uniform sampling.

Note that as we already know, in some sampling combinations such as sampling all outputs, every $M\pi/NB$, where M/N is not an integer, unique reconstruction is not possible. In these cases, the matrix $\boldsymbol{T}(\omega)$ is always noninvertible, no matter what system $\boldsymbol{H}(\omega)$ is used.

From the analysis done for the expansion by an integer factor, we realize that the general rule for unique solution is that equations should not be wasted. This general rule leads to the following two equivalent *necessary* conditions for unique reconstruction.

i) For any possible set of equations, the number of equations in the set must be less than or equal to the number of all the unknowns appearing in these equations.

ii) For any possible set of intervals (each representing N unknowns), the number of unknowns in the set (which is N times the number of intervals in the set) must be less than or equal to the number of all equations in which one or more of these intervals appear.

Condition ii) is clear. If we have a set of K unknowns that appear only in $L < K$ equations, we cannot have a unique solution for these unknowns. Similarly, condition i) is clear. Since the total number of equations and unknowns is the same mN, then if we have a set of L equations in which there are only $K < L$ unknowns, we must have, in the rest of the $(mN-L)$ equations, $(mN-K) > (mN-L)$ unknowns, for which we cannot have a unique solution.

In addition to these conditions, for unique reconstruction, the matrix $\boldsymbol{H}(\omega)$ should be such that the determinant of the resulting matrix $\boldsymbol{T}(\omega)$ is not zero for every $\omega \in [-B, -B+c]$. Intuitively, in this case, the samples of the M output channels are considered to be "independent." Such a matrix $\boldsymbol{H}(\omega)$ can always be found when the conditions above are satisfied, and therefore, these conditions are essentially necessary and sufficient in the sense that this sampling combination allows a unique reconstruction. Note that if the conditions above are not satisfied, then for any choice of $\boldsymbol{H}(\omega)$, the determinant of $\boldsymbol{T}(\omega)$ is zero.

Unfortunately, the benefit of the conditions above can sometimes be limited. It is true that in some cases, we can verify immediately that the necessary conditions are not satisfied and rule out a specific sampling combination. However, if the sampling combination allows a unique reconstruction, we would have to check all 2^m possible sets of intervals to see that neither choice violated the second condition above. A better way, in this case, would be to find the determinant of $\boldsymbol{T}(\omega)$, which requires $O((mN)^3)$ operations.

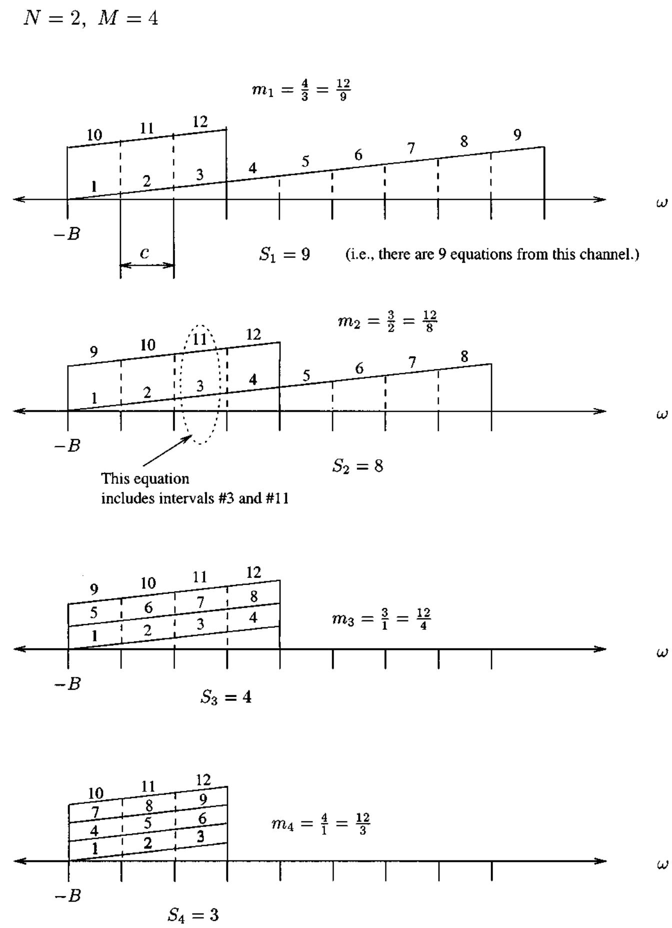

Fig. 4. Intervals and unknowns in nonequal uniform sampling.

There is yet another possibility. It is shown in the Appendix that determining whether a unique reconstruction is possible, i.e., whether the matrix $\boldsymbol{T}(\omega)$ is invertible for all $\omega \in [-B, -B+c]$, is equivalent to the problem of perfect matching in a bipartite graph. The graph nodes are the rows and the columns of the $(mN) \times (mN)$ matrix $\boldsymbol{T}(\omega)$, and the edges correspond the nonzero elements of $\boldsymbol{T}(\omega)$ that represent the mN equations of mN unknowns described above. The perfect matching is a well-known combinatorial problem. This problem is equivalent to the Hall *marriage problem*, to which Hall [12], [13] provided necessary and sufficient conditions. These conditions can be verified using the "Hungarian method" of König and Egerváry, which requires $O((mN)^3)$ operations, or even by $O((mN)^{5/2})$ operations of Hopcroft and Karp [14] or Even [15]. This approach has the additional advantage that the question of whether or not unique reconstruction is possible can be answered without choosing specific values for the MIMO system $\boldsymbol{H}(\omega)$.

Although we cannot specify explicitly the exact condition for allowing reconstruction that can be checked easily, we provide below a simple *sufficient condition*. This condition implies that when all m_k's are integers, we can have a unique reconstruction.

A. A Sufficient Condition for Unique Reconstruction

We start with the set R consisting of all the intervals $1, 2, \cdots, m$. For this set, the number of equations is mN, and the number of unknowns is also mN. We now split this set into two sets R and Q having $(m-q)$ and q intervals, respectively. We denote the number of all equations including *only* intervals belonging to the set Q as N_Q. Let us now find an upper limit for N_Q. It is clear that in a channel where $m_k \geq q+1$, there are at least $q+1$ replicas, i.e., $q+1$ intervals, in every equation. Therefore, in all output channels having $m_k \geq q+1$, we have no equations including only intervals from the set Q.

In a channel with $m_k < q+1$, we have at least $\lfloor m_k \rfloor$ replicas in all the equations. This means that it might be possible to choose such q intervals so that we will have $\lfloor q/\lfloor m_k \rfloor \rfloor$ equations, which includes only intervals belonging to the set Q. Therefore

$$N_Q \leq \sum_{m_k < q+1} \left\lfloor \frac{q}{\lfloor m_k \rfloor} \right\rfloor. \tag{31}$$

This is an inequality because there is no guarantee that the q intervals chosen for the set Q will fit all, or even any, equation in all channels having $m_k < q+1$.

The condition

$$\sum_{m_k < q+1} \left\lfloor \frac{q}{\lfloor m_k \rfloor} \right\rfloor \leq qN; \qquad 1 \leq q \leq m-1 \tag{32}$$

is a sufficient condition allowing a unique solution. This is so because N_R, which denotes the number of remaining equations after removing the N_Q equations that include only intervals that belong to the set Q, satisfies

$$\begin{aligned} N_R &= mN - N_Q \\ &\geq mN - \sum_{m_k < q+1} \left\lfloor \frac{q}{\lfloor m_k \rfloor} \right\rfloor \\ &\geq mN - qN \\ &= (m-q)N \end{aligned} \tag{33}$$

which is actually condition ii) mentioned above.

Suppose that all m_k's are integers. Therefore, we have

$$\begin{aligned} \sum_{m_k < q+1} \left\lfloor \frac{q}{\lfloor m_k \rfloor} \right\rfloor &= \sum_{m_k < q+1} \left\lfloor \frac{q}{m_k} \right\rfloor \\ &\leq \sum_{m_k < q+1} \frac{q}{m_k} \\ &\leq \sum_{k=1}^{M} \frac{q}{m_k} = q \sum_{k=1}^{M} \frac{1}{m_k} = qN. \end{aligned} \tag{34}$$

This implies that choosing integer m_k's allows a unique reconstruction.

The fact that unique reconstruction is possible for integer m_k's can be shown in another way. In this case, we can choose $\boldsymbol{H}(\omega)$ so that one of the representations of $\boldsymbol{T}(\omega)$ is a doubly stochastic matrix. A doubly stochastic matrix is a convex combination of permutation matrices, and therefore, a perfect match is possible in the bipartite graph associated with the matrix $\boldsymbol{T}(\omega)$. As noted above, this means that unique reconstruction is possible.

An example showing that (32) is not a necessary condition is the case of $N=1$, $M=2$, where $m_1 = 8/5$, and $m_2 = 8/3$. Here, we see that (32) is not satisfied, e.g., for $q=2$. However, calculating the determinant of the appropriate matrix $\boldsymbol{T}$ shows that a unique solution is possible.

B. Interpolation Formula for Integer m_k's

Integer m_k's allow reconstruction. We now present the associated interpolation formula. We first note that for every $M > N$, we can always choose $m_k = 1$ for $k = 1, \cdots, N-1$ and $m_k = M - N + 1$ for $k = N, \cdots, M$. Since these choices satisfy (30), it means that we always have at least one sampling combination allowing reconstruction in which the sampling periods are integer multiples of π/B for every $M \geq N$.

When sampling at $1/m_k$, the Nyquist rate, i.e., at a sampling period of $T_k = m_k \pi/B = m_k T$ where m_k is an integer, we get aliased versions of the output signals, which, at the frequency domain, are periodic with a period $c_k = 2B/m_k$. We could perform the same analysis as in the integer expansion factor case. However, such an analysis leads to a relatively complicated interpolation formula. Thus, we use a different approach that results in a much simpler interpolation formula. Instead of sampling the kth output channel every T_k, we stagger the kth output, i.e., first split it into $s_k = m/m_k$ identical channels and then shift backward the ith duplicated signal by $(i-1)T_k = (i-1)m_k(\pi/B)$. We therefore get a modified MIMO system having $mN = \sum_{k=1}^{M} s_k$ outputs, as in Fig. 5. Sampling each output signal of the modified MIMO system at sampling period $mT = m\pi/B$ is equivalent to sampling the kth output channel every T_k. This modified system has N inputs and mN outputs. Therefore, we get an integer expansion factor, for which we had already found the interpolation formulas; see (9), (12), and (14).

Let us now describe the system of equations resulting from the modification of the MIMO system mentioned above. First, we denote by $\boldsymbol{G}^a(\omega)$ the mN-dimensional vector given by

$$\begin{aligned} \boldsymbol{G}^a(\omega)^T = [&G^a_{1,1}(\omega), G^a_{1,2}(\omega), \cdots, G^a_{1,s_1}(\omega), G^a_{2,1}(\omega) \\ &G^a_{2,2}(\omega), \cdots, G^a_{2,s_2}(\omega), \cdots, G^a_{M,1}(\omega) \\ &G^a_{M,2}(\omega), \cdots, G^a_{M,s_M}(\omega)] \end{aligned} \tag{35}$$

i.e., its lth component is $G^a_{1,l}(\omega)$ for $l \leq s_1$, $G^a_{2,l-s_1}(\omega)$ for $s_1 < l \leq s_1 + s_2$ and $G^a_{k,l-v_k}(\omega)$ for $v_k < l \leq v_k + s_k$, where

$$v_k = \sum_{i=0}^{k-1} s_i; \qquad s_0 = 0 \tag{36}$$

and $G^a_{k,l}(\omega)$ is the Fourier transform of $g_k(t + (l-1)T_k)$ when it is sampled every mT ($l = 1, \cdots, s_k$).

Then, we write

$$\boldsymbol{G}^a(\omega) = \frac{c}{2\pi} \boldsymbol{T}(\omega) \boldsymbol{F}^a(\omega) \qquad \omega \in [-B, -B+c] \tag{37}$$

where $\boldsymbol{F}^a(\omega)$ is identical to that of (5) and where $\boldsymbol{T}(\omega)$ is an $mN \times mN$ matrix whose (i, q)th component is given by (38)

$$\begin{aligned} T_{iq}(\omega) = &H_{k, \lceil q/m \rceil}(\omega + [(q-1) \bmod m] \cdot c) \\ &\cdot e^{j(\omega + [(q-1) \bmod m] \cdot c)(i - v_k - 1)T_k} \\ &v_k < i \leq v_k + s_k, \; \omega \in [-B, -B+c]. \end{aligned} \tag{38}$$

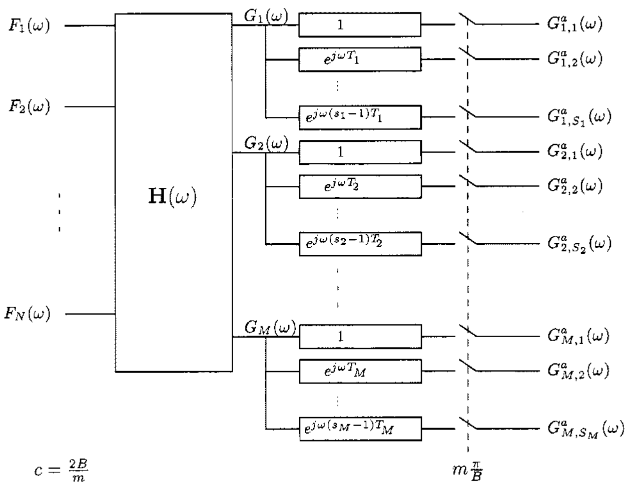

Fig. 5. Modifying the system for integer m_k's.

Thus, $\boldsymbol{T}(\omega)$ is made of $M \times N$ blocks of $s_k \times m$ each. The (k, l)th block is described in (39), shown at the bottom of the page.

Note that if $\det\{\boldsymbol{T}(\omega)\} \neq 0$ for $\omega \in [-B, -B + c]$, we can reconstruct the N input signals from the samples of the M output channels of the original MIMO system. According to (9), (12), and (14), we can find the $y_{i,q'}(t)$'s, where $q' = 1, \cdots, mN$. The interpolation formula is

$$\begin{aligned} f_i(t) &= \sum_{q'=1}^{mN} \left[\sum_{n=-\infty}^{\infty} g_{q'}(nmT) y_{i,q'}(t - nmT) \right] \\ &= \sum_{k=1}^{M} \sum_{q=1}^{s_k} \left[\sum_{n=-\infty}^{\infty} g_k((q-1)T_k + nmT) \right. \\ &\qquad \left. \cdot\, y_{i,v_k+q}(t - nmT) \right] \\ &= \sum_{k=1}^{M} \sum_{q=1}^{s_k} \left[\sum_{n=-\infty}^{\infty} g_k((q-1)T_k + ns_kT_k) \right. \\ &\qquad \left. \cdot\, y_{i,v_k+q}(t - ns_kT_k) \right] \\ &= \sum_{k=1}^{M} \left[\sum_{n=-\infty}^{\infty} g_k(nT_k) \right. \\ &\qquad \left. \cdot\, y_{i,v_k+(n \bmod s_k)+1}\,(t + (n \bmod s_k)T_k - nT_k) \right]. \end{aligned} \tag{40}$$

In order to get a simple interpolation formula that resembles (14), we define

$$y_{i,k}(t, n) = y_{i,v_k+(n \bmod s_k)+1}(t + (n \bmod s_k)T_k) \tag{41}$$

and then we can write the interpolation formula

$$f_i(t) = \sum_{k=1}^{M} \left[\sum_{n=-\infty}^{\infty} g_k(nT_k) y_{i,k}(t - nT_k, n) \right]. \tag{42}$$

C. An Example

Using the condition of (32), we can easily verify that the sampling combination of $N = 2$, $M = 3$, and $m_1 = m$, $m_2 = m/(m-1)$, and $m_3 = 1$ allows a unique solution for every integer m, which is greater than or equal to two. The following example for the case of $N = 2$ and $M = 3$ demonstrates that we can apply the interpolation method described in the previous section to reconstruct the input signals, although the m_k's are not integers. The blocks described in (39) will still be of size $s_k \times m$, where $s_k = m/m_k$.

In our example, we have $\boldsymbol{H}(\omega)$

$$\boldsymbol{H}(\omega) = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} \cdot e^{j\omega T} & H_{22} \cdot e^{j\omega T} \\ H_{21} & H_{22} \end{bmatrix} \tag{43}$$

where H_{11}, H_{12}, H_{21}, and H_{22} are constants, and $T = \pi/B$.

We choose to sample the first output $g_1(t)$ every T, the second output $g_2(t)$ every $3T$, and the third output $g_3(t)$ every $(3/2)T$, i.e., $m_1 = 1$, $m_2 = 3$, and $m_3 = 3/2$. In this case, we have $m = 3$ and $c = 2B/3$.

According to (38), $\boldsymbol{T}(\omega)$ is given by (44), shown at the bottom of the next page. From (9), (12), and (44) we find (44a), also shown at the bottom of the next page, where

$$K_1 = \frac{H_{22}}{H_{11}H_{22} - H_{12}H_{21}}$$
$$K_2 = \frac{-H_{12}}{H_{11}H_{22} - H_{12}H_{21}}.$$

Note that the right-hand side in the first three equations in (44a) is the Sinc function $\sin(Bt)/Bt$ multiplied by K_1 and centered at 0, T, and $2T$, respectively. The right-hand side in the last three equations equals Yen's [16] reconstruction filters for periodic nonuniform sampling with $T_1 = T$, $T_2 = 0$, and $T_3 = (3/2)T$, multiplied by K_2. This is not surprising since the modified system in this specific example can be represented by a 2×2 matrix of

$$\begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix}$$

$$\begin{bmatrix} H_{kl}(\omega) & H_{kl}(\omega + c) & \cdots & H_{kl}(\omega + (m-1)c) \\ H_{kl}(\omega)e^{j\omega T_k} & H_{kl}(\omega + c)e^{j(\omega+c)T_k} & \cdots & H_{kl}(\omega + (m-1)c)e^{j(\omega+(m-1)c)T_k} \\ H_{kl}(\omega)e^{j\omega 2T_k} & H_{kl}(\omega + c)e^{j(\omega+c)2T_k} & \cdots & H_{kl}(\omega + (m-1)c)e^{j(\omega+(m-1)c)2T_k} \\ \vdots & \vdots & & \vdots \\ H_{kl}(\omega)e^{j\omega(s_k-1)T_k} & H_{kl}(\omega + c)e^{j(\omega+c)(s_k-1)T_k} & \cdots & H_{kl}(\omega + (m-1)c)e^{j(\omega+(m-1)c)(s_k-1)T_k} \end{bmatrix} \tag{39}$$

followed by uniform sampling of one output and periodic nonuniform sampling of the other output. Therefore, the Sinc function reconstructs the first output, Yen's functions reconstruct the other output, and then, we use the inverse of the matrix to reconstruct the two inputs, which is equal to the multiplications by the factors mentioned above. The reconstruction filters for the second input are similar but with the factors of

$$\frac{-H_{21}}{H_{11}H_{22}-H_{12}H_{21}}$$

and

$$\frac{H_{11}}{H_{11}H_{22}-H_{12}H_{21}}.$$

Without using the combination suggested in (41), the interpolation formula is given by

$$\begin{aligned} f_1(t) = &\sum_{n=-\infty}^{\infty} g_1(n3T)y_{1,1}(t-n3T) \\ &+ \sum_{n=-\infty}^{\infty} g_1(T+n3T)y_{1,2}(t-n3T) \\ &+ \sum_{n=-\infty}^{\infty} g_1(2T+n3T)y_{1,3}(t-n3T) \\ &+ \sum_{n=-\infty}^{\infty} g_2(n3T)y_{1,4}(t-n3T) \\ &+ \sum_{n=-\infty}^{\infty} g_3(n\frac{3}{2}T)y_{1,5}(t-n3T) \\ &+ \sum_{n=-\infty}^{\infty} g_3\left(\frac{3}{2}T+n\,\frac{3}{2}\,T\right)y_{1,6}(t-n3T). \end{aligned} \tag{45}$$

Using (41) we find

$$y_{1,1}(t,\,n) = K_1\,\frac{\sin(Bt)}{Bt} \tag{46}$$

$$y_{1,2}(t,\,n) = K_2\,\frac{\sin\left(\frac{B}{3}\,(t-T)\right)}{\left(\frac{B}{3}\,(t-T)\right)}\,\frac{\sin\left(\frac{2B}{3}\,t\right)}{\cos\left(\frac{BT}{6}\right)} \tag{47}$$

$$y_{1,3}(t,\,n) = K_2\,\frac{\sin\left(\frac{B}{3}\,t\right)}{\left(\frac{B}{3}\,t\right)}\left[\frac{\cos\left(\frac{B}{3}\left(2t-\frac{T}{2}\right)\right)}{2\,\cos\left(\frac{BT}{6}\right)} + \frac{1}{2} - (-1)^n\,\frac{\sin\left(\frac{2B}{3}\,t\right)}{2\,\cos\left(\frac{BT}{6}\right)}\right] \tag{48}$$

$$\boldsymbol{T}(\omega) = \begin{bmatrix} H_{11} & H_{11} & H_{11} & H_{12} & H_{12} & H_{12} \\ H_{11}\cdot e^{j\omega T} & H_{11}\cdot e^{j(\omega+c)T} & H_{11}\cdot e^{j(\omega+2c)T} & H_{12}\cdot e^{j\omega T} & H_{12}\cdot e^{j(\omega+c)T} & H_{12}\cdot e^{j(\omega+2c)T} \\ H_{11}\cdot e^{j2\omega T} & H_{11}\cdot e^{j2(\omega+c)T} & H_{11}\cdot e^{j2(\omega+2c)T} & H_{12}\cdot e^{j2\omega T} & H_{12}\cdot e^{j2(\omega+c)T} & H_{12}\cdot e^{j2(\omega+2c)T} \\ H_{21}\cdot e^{j\omega T} & H_{21}\cdot e^{j(\omega+c)T} & H_{21}\cdot e^{j(\omega+2c)T} & H_{22}\cdot e^{j\omega T} & H_{22}\cdot e^{j(\omega+c)T} & H_{22}\cdot e^{j(\omega+2c)T} \\ H_{21} & H_{21} & H_{21} & H_{22} & H_{22} & H_{22} \\ H_{21}\cdot e^{j(3/2)\omega T} & H_{21}\cdot e^{j(3/2)(\omega+c)T} & H_{21}\cdot e^{j(3/2)(\omega+2c)T} & H_{22}\cdot e^{j(3/2)\omega T} & H_{22}\cdot e^{j(3/2)(\omega+c)T} & H_{22}\cdot e^{j(3/2)(\omega+2c)T} \end{bmatrix} \tag{44}$$

$$\begin{aligned} &y_{1,1}(t) = K_1\,\frac{\sin(Bt)}{Bt} \qquad y_{1,2}(t) = K_1\,\frac{\sin(B(t-T))}{B(t-T)} \qquad y_{1,3}(t) = K_1\,\frac{\sin(B(t-2T))}{B(t-2T)} \\ &y_{1,4}(t) = K_2\,\frac{\sin\left(\frac{B}{3}(t-T)\right)\sin\left(\frac{B}{3}(t-0)\right)\sin\left(\frac{B}{3}\left(t-\frac{3}{2}T\right)\right)}{\sin\left(\frac{B}{3}(T-0)\right)\sin\left(\frac{B}{3}\left(T-\frac{3}{2}T\right)\right)\frac{B}{3}\,(t-T)} \\ &y_{1,5}(t) = K_2\,\frac{\sin\left(\frac{B}{3}\,(t-T)\right)\sin\left(\frac{B}{3}\,(t-0)\right)\sin\left(\frac{B}{3}\left(t-\frac{3}{2}T\right)\right)}{\sin\left(\frac{B}{3}\,(0-T)\right)\sin\left(\frac{B}{3}\left(0-\frac{3}{2}T\right)\right)\frac{B}{3}\,(t-0)} \\ &y_{1,6}(t) = K_2\,\frac{\sin\left(\frac{B}{3}\,(t-T)\right)\sin\left(\frac{B}{3}\,(t-0)\right)\sin\left(\frac{B}{3}\left(t-\frac{3}{2}T\right)\right)}{\sin\left(\frac{B}{3}\left(\frac{3}{2}T-T\right)\right)\sin\left(\frac{B}{3}\left(\frac{3}{2}T-0\right)\right)\frac{B}{3}\left(t-\frac{3}{2}T\right)} \end{aligned} \tag{44a}$$

and therefore, we get the final interpolation formula

$$
\begin{aligned}
f_1(t) = &\sum_{n=-\infty}^{\infty} g_1(nT) \cdot y_{1,1}(t - nT, n) \\
&+ \sum_{n=-\infty}^{\infty} g_2(n3T) \cdot y_{1,2}(t - n3T, n) \\
&+ \sum_{n=-\infty}^{\infty} g_3\left(n\frac{3}{2}T\right) \cdot y_{1,3}\left(t - n\frac{3}{2}T, n\right)
\end{aligned} \tag{49}
$$

which of course is equal to (45).

IV. Equal Periodic Nonuniform Sampling

Another possible solution to the case where M/N is not an integer, is to employ nonuniform sampling. The simplest scheme is to sample each of the M outputs at the Nyquist rate, but in each of the channels, use only the first N samples from every set of M samples and delete the remaining $M - N$ samples, as depicted in Fig. 6.

Similarly to Section III-B, we first split every output channel to N identical signals and then shift backward (in time) the ith duplicated signal by $(i-1)T = (i-1)\pi/B$. We therefore get a modified MIMO system having MN outputs. If we now sample each output signal of the modified MIMO system at sampling period $MT = M(\pi/B)$, it is equivalent to sampling every original output channel N samples at Nyquist rate (sampling period of $T = \pi/B$) and then waiting for a period of $M - N$ samples before sampling again, which means an average sampling rate of N/M times the Nyquist rate. This modified system has N inputs and MN outputs. Again, as in Section III-B, we reached a situation of an integer expansion factor, for which we had already found the interpolation formula.

Let us now describe the system of equations for the modified MIMO system. We denote by $\boldsymbol{G}^a(\omega)$ the MN-dimensional vector given by

$$
\begin{aligned}
\boldsymbol{G}^a(\omega)^T = [&G^a_{1,1}(\omega), G^a_{1,2}(\omega), \cdots, G^a_{1,N}(\omega), G^a_{2,1}(\omega) \\
&G^a_{2,2}(\omega), \cdots, G^a_{2,N}(\omega), \cdots, G^a_{M,1}(\omega) \\
&G^a_{M,2}(\omega), \cdots, G^a_{M,N}(\omega)]
\end{aligned} \tag{50}
$$

i.e., its lth component is $G^a_{\lceil l/N \rceil, (l-1) \bmod N+1}(\omega)$, where $G^a_{k,l}(\omega)$ is the Fourier transform of $g_k(t + (l-1)T)$ when it is sampled every MT and $l = 1, \cdots, N$.

We also denote by $\boldsymbol{F}^a(\omega)$ the MN-dimensional vector given by

$$
\begin{aligned}
\boldsymbol{F}^a(\omega)^T = [&F_1(\omega), F_1(\omega + c), \cdots, F_1(\omega + (M-1)c) \\
&\cdots, F_N(\omega), \cdots, F_N(\omega + (M-1)c)]
\end{aligned} \tag{51}
$$

where $c = 2B/M$. Therefore, we have

$$
\boldsymbol{G}^a(\omega) = \frac{c}{2\pi} \boldsymbol{T}(\omega) \boldsymbol{F}^a(\omega) \qquad \omega \in [-B, -B + c] \tag{52}
$$

where $\boldsymbol{T}(\omega)$ is an $MN \times MN$ matrix whose (i, q)th component is given by

$$
\begin{aligned}
T_{iq}(\omega) = &H_{\lceil i/N \rceil, \lceil q/M \rceil}(\omega + [(q-1) \bmod M] \cdot c) \\
&\cdot e^{j(\omega + [(q-1) \bmod M] \cdot c)[(i-1) \bmod N]\pi/B}
\end{aligned}
$$

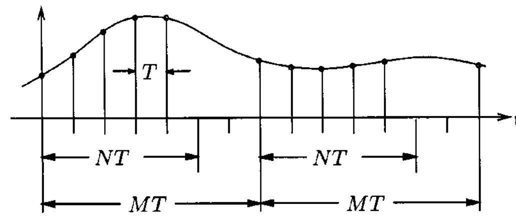

Fig. 6. Simple periodic nonuniform sampling.

$$
\omega \in [-B, -B + c] \tag{53}
$$

i.e., $\boldsymbol{T}(\omega)$ is made of $M \times N$ blocks of $N \times M$ each, where the (k, l)th block is described in (54), shown at the bottom of the next page. When $\det\{\boldsymbol{T}(\omega)\} \neq 0$ for $\omega \in [-B, -B + c]$, we can uniquely reconstruct the N input signals from the nonuniform but periodic samples of the M original output channels.

Let us now use (9), (12), and (14) to find the $y_{i,q'}(t)$'s, where $q' = 1, \cdots, MN$. The interpolation formula is therefore described by

$$
\begin{aligned}
f_i(t) &= \sum_{q'=1}^{MN} \left[\sum_{n=-\infty}^{\infty} g_{q'}(nMT) y_{i,q'}(t - nMT) \right] \\
&= \sum_{k=1}^{M} \sum_{q=1}^{N} \left[\sum_{n=-\infty}^{\infty} g_k((q-1)T + nMT) \right. \\
&\qquad \left. \cdot\, y_{i,(k-1)N+q}(t - nMT) \right] \\
&= \sum_{k=1}^{M} \left[\sum_{n=-\infty}^{\infty} g_k\left((n \bmod N)T + \left\lfloor \frac{n}{N} \right\rfloor MT\right) \right. \\
&\qquad \left. \cdot\, y_{i,(k-1)N+(n \bmod N)+1}\left(t - \left\lfloor \frac{n}{N} \right\rfloor MT\right) \right].
\end{aligned} \tag{55}
$$

In order to get a simple interpolation formula that resembles (14), we should unite the N $y_{i,q}(t)$'s for $(k-1)N < q \leq kN$. We now define

$$
y_{i,k}(t, n) = y_{i,(k-1)N+(n \bmod N)+1}(t) \tag{56}
$$

and denote

$$
g_k[n] = g_k\left((n \bmod N)T + \left\lfloor \frac{n}{N} \right\rfloor MT\right). \tag{57}
$$

Using this, we can write the interpolation formula

$$
f_i(t) = \sum_{k=1}^{M} \left[\sum_{n=-\infty}^{\infty} g_k[n] y_{i,k}\left(t - \left\lfloor \frac{n}{N} \right\rfloor MT, n\right) \right]. \tag{58}
$$

A similar analysis can be performed when the N sampling points are located arbitrarily along the interval MT at locations T_i, where $0 \leq T_i < MT$, and $i = 1, 2, \cdots, N$. If this is the case, the only modification in the matrix $\boldsymbol{T}(\omega)$ is that the blocks described in (54) will be as in (59), shown at the bottom of the next page.

It can be easily verified that when the T_i's are chosen to be $(M/N)T = (M/N)(\pi/B)$, i.e., uniform sampling, the resulting matrix $\boldsymbol{T}(\omega)$ is noninvertible when M/N is not an integer.

V. NOISE SENSITIVITY OF VSE SYSTEMS

The analysis conducted so far assumed that the sample values were known within an infinite precision. In practice, we never have the exact values of the samples due to quantization and noise; therefore, it is interesting to explore the sensitivity of VSE systems.

The first issue is well posedness. In GSE, this problem was initially discussed by Cheung and Marks [17], who found a sufficient condition for ill-posedness of the system. Under their definition, an ill-posed GSE system produces a reconstruction error with unbounded variance when a bounded variance noise is added to the samples. Later on, Brown and Cabrera [18], [19] found a necessary and sufficient condition for well posedness of GSE systems. We find a similar condition for VSE, where in our definition, a well posed VSE system produces a reconstruction error with bounded variance in all N reconstructed signals when a bounded variance noise is added to the samples. In deriving this condition, we also get an expression for the reconstruction noise power for each reconstructed signal. This result is discussed in Section V-A.

The expression for the reconstruction noise level provides a quantitative measure for the noise sensitivity of the system. Using this, we can then determine the optimal VSE systems, i.e., the systems that minimize the total reconstruction noise level, under some power constraints. This is discussed in Section V-B.

A. Reconstruction Noise Power and Well Posedness of VSE Systems

In our analysis, the noise of the VSE system at reconstruction is a result of adding a zero mean white discrete stochastic noise sequence to each sample sequence, generated by sampling the M output channels, prior to reconstruction. This noise represents the quantization errors and other system inaccuracies. It is assumed that the quantization noise is statistically independent of the signals and that noise sample sequences added to different channels are also statistically independent. Denote the nth noise value, which is added to the nth sample of the kth channel by $v_k(nT)$, where

$$E\{v_k(nT)v_q^*(mT)\} = \sigma_v^2\delta_{k,\,q}\delta_{n,\,m} \tag{60}$$

and $v_k(nT)$ is a zero mean random variable.

We now calculate the contribution of this noise to the reconstructed signal $f_i^r(t)$ produced by

$$f_i^r(t) = \sum_{k=1}^{M}\left[\sum_{n=-\infty}^{\infty}(g_k(nT)+v_k(nT))y_{i,\,k}(t-nT)\right]. \tag{61}$$

We specifically want to find the value of $E\{|v_i^r(t)|^2\}$, where

$$v_i^r(t) = f_i^r(t) - f_i(t) = \sum_{k=1}^{M}\left[\sum_{n=-\infty}^{\infty} v_k(nT)y_{i,\,k}(t-nT)\right]. \tag{62}$$

However, the signal $v_i^r(t)$ is not a wide sense stationary (WSS) signal since due to aliasing, it is a sum of m correlated, WSS signals, shifted in frequency. Thus, we look for the time-averaged value $\overline{E\{|v_i^r(t)|^2\}}$.

Instead of analyzing the time domain expression of the reconstruction noise (reconstruction error) given by (62), we conduct our analysis in the frequency domain. Using (4), we can write the reconstruction equation in the frequency domain

$$\boldsymbol{F}^a(\omega) = \frac{2\pi}{c}\boldsymbol{T}(\omega)^{-1}\boldsymbol{G}^a(\omega) \qquad \omega \in [-B,\, -B+c]. \tag{63}$$

As seen from (63), every row of the matrix describes a "slice" of the reconstructed spectrum. The $[m(i-1)+k]$th row describes the reconstruction of $F_i(\omega+(k-1)c)$ for $\omega \in [-B,\, -B+c]$, which means it describes $F_i(\omega)$ for $\omega \in [-B+(k-1)c,\, -B+kc]$.

We now analyze the effect of adding the uncorrelated noise samples $v_l(nT)$ to the lth output of the MIMO system. This is equivalent to adding to $g_l(t)$, prior to sampling, a WSS stochastic process $v_l(t)$, which is bandlimited to $\omega \in [-c/2,\, c/2]$ and has a spectral power density of $S_{v_l v_l}(\omega) = N_0$, where

$$\begin{bmatrix} H_{kl}(\omega) & H_{kl}(\omega+c) & \cdots & H_{kl}(\omega+(M-1)c) \\ H_{kl}(\omega)e^{j\omega T} & H_{kl}(\omega+c)e^{j(\omega+c)T} & \cdots & H_{kl}(\omega+(M-1)c)e^{j(\omega+(M-1)c)T} \\ H_{kl}(\omega)e^{j\omega 2T} & H_{kl}(\omega+c)e^{j(\omega+c)2T} & \cdots & H_{kl}(\omega+(M-1)c)e^{j(\omega+(M-1)c)2T} \\ H_{kl}(\omega)e^{j\omega(N-1)T} & H_{kl}(\omega+c)e^{j(\omega+c)(N-1)T} & \cdots & H_{kl}(\omega+(M-1)c)e^{j(\omega+(M-1)c)(N-1)T} \end{bmatrix} \tag{54}$$

$$\begin{bmatrix} H_{kl}(\omega)e^{j\omega T_1} & H_{kl}(\omega+c)e^{j(\omega+c)T_1} & \cdots & H_{kl}(\omega+(M-1)c)e^{j(\omega+(M-1)c)T_1} \\ H_{kl}(\omega)e^{j\omega T_2} & H_{kl}(\omega+c)e^{j(\omega+c)T_2} & \cdots & H_{kl}(\omega+(M-1)c)e^{j(\omega+(M-1)c)T_2} \\ H_{kl}(\omega)e^{j\omega T_3} & H_{kl}(\omega+c)e^{j(\omega+c)T_3} & \cdots & H_{kl}(\omega+(M-1)c)e^{j(\omega+(M-1)c)T_3} \\ H_{kl}(\omega)e^{j\omega T_N} & H_{kl}(\omega+c)e^{j(\omega+c)T_N} & \cdots & H_{kl}(\omega+(M-1)c)e^{j(\omega+(M-1)c)T_N} \end{bmatrix} \tag{59}$$

$N_0 = T\sigma_v^2$ and $T = M\pi/NB$, as described in Fig. 7. The power spectrum of the noise at the output of the system represented by the $[m(i-1)+k]$th row of $\boldsymbol{T}(\omega)^{-1}$, resulting from $v_l(t)$, is given by

$$S_{v_i^r v_i^r}(\omega+(k-1)c)_l = \frac{N_0}{T^2}\left(\frac{2\pi}{c}\right)^2 \left|T_{[m(i-1)+k],\,l}(\omega)^{-1}\right|^2 \tag{64}$$

where $\omega+(k-1)c$ implies that this contribution is in the interval $\omega \in [-B+(k-1)c,\, -B+kc]$, the subscript l implies that this is the contribution of the noise coming from the lth channel only, and the subscript i implies that this is the contribution to the ith reconstructed signal. We now wish to calculate the joint contribution of all M noise sources to the ith reconstructed signal. Because the noise sources are statistically independent (and therefore uncorrelated), the total contribution is simply the sum of all separate contributions $S_{v_i^r v_i^r}(\omega+(k-1)c)_l$

$$\begin{aligned} S_{v_i^r v_i^r}&(\omega+(k-1)c) \\ &= \sum_{l=1}^{M} S_{v_i^r v_i^r}(\omega+(k-1)c)_l \\ &= \frac{N_0}{T^2}\left(\frac{2\pi}{c}\right)^2 \sum_{l=1}^{M}\left|T_{[m(i-1)+k],\,l}(\omega)^{-1}\right|^2. \end{aligned} \tag{65}$$

This is the spectrum of a frequency slice of the reconstruction noise $v_i^r(t)$ imposed on the ith reconstructed signal, which is shifted left in frequency by $(k-1)c$. We denote this frequency shifted left slice by $v_i^r(t)^{(k)}$. Note that although $v_i^r(t)$ is not a WSS signal, its m components $v_i^r(t)^{(k)}$ are WSS. We now calculate $E\{|v_i^r(t)|^2\}$. The signal $v_i^r(t)$ is given by

$$\begin{aligned} v_i^r(t) &= \sum_{k=1}^{m} v_i^r(t)^{(k)} e^{j(K-1)ct} \\ &= \boldsymbol{E}(t)^T \boldsymbol{v}_i^r(t) \end{aligned} \tag{66}$$

where

$$\boldsymbol{v}_i^r(t)^T = \left[v_i^r(t)^{(1)},\, v_i^r(t)^{(2)},\, \cdots,\, v_i^r(t)^{(m)}\right] \tag{67}$$

and

$$\boldsymbol{E}(t)^T = \left[1,\, e^{jct},\, e^{j2ct},\, \cdots,\, e^{j(m-1)ct}\right] \tag{68}$$

and so have

$$\begin{aligned} E\left\{|v_i^r(t)|^2\right\} &= E\left\{\boldsymbol{E}(t)^T \boldsymbol{v}_i^r(t)\boldsymbol{v}_i^r(t)^{T^*}\boldsymbol{E}(t)^*\right\} \\ &= \sum_{k=1}^{m}\sum_{l=1}^{m} [\boldsymbol{R}_{\boldsymbol{v}_i^r\boldsymbol{v}_i^r}(0)]_{k,\,l} e^{j(k-l)ct} \end{aligned} \tag{69}$$

where $\boldsymbol{R}_{\boldsymbol{v}_i^r\boldsymbol{v}_i^r}(\tau)$ is the correlation matrix of the vector $\boldsymbol{v}_i^r(t)$. Since e^{jkct} is periodic in t with an integer number of periods in T, we have

$$\begin{aligned} &\overline{E\left\{|v_i^r(t)|^2\right\}} \\ &= \frac{1}{T}\int_0^T E\left\{|v_i^r(t)|^2\right\} dt \\ &= \frac{1}{T}\int_0^T \sum_{k=1}^{m}\sum_{l=1}^{m} [\boldsymbol{R}_{\boldsymbol{v}_i^r\boldsymbol{v}_i^r}(0)]_{k,\,l} e^{j(k-l)ct}\, dt \\ &= \sum_{k=1}^{m} [\boldsymbol{R}_{\boldsymbol{v}_i^r\boldsymbol{v}_i^r}(0)]_{k,\,k} \\ &= \sum_{k=1}^{m} \frac{1}{2\pi}\int_{-B}^{-B+c} S_{v_i^r v_i^r}(\omega+(k-1)c)\, d\omega \\ &= \sum_{k=1}^{m} \frac{2\pi N_0}{c^2T^2}\int_{-B}^{-B+c} \sum_{l=1}^{M}\left|T_{[m(i-1)+k],\,l}(\omega)^{-1}\right|^2 d\omega. \end{aligned} \tag{70}$$

Using the relations $N_0 = T\sigma_v^2, T = M\pi/NB$, and $c = 2B/m$, we find that

$$\begin{aligned} &\overline{E\left\{|v_i^r(t)|^2\right\}} \\ &\quad = \frac{\sigma_v^2}{c}\int_{-B}^{-B+c}\sum_{k=1}^{m}\sum_{l=1}^{M}\left|T_{[m(i-1)+k],\,l}(\omega)^{-1}\right|^2 d\omega. \end{aligned} \tag{71}$$

We now denote the matrix composed of the rows $[m(i-1)+1], \cdots, mi$ of $\boldsymbol{T}(\omega)^{-1}$ by $\boldsymbol{T}(\omega)_i^{-1}$ so that (71) may be written as

$$\overline{E\left\{|v_i^r(t)|^2\right\}} = \frac{\sigma_v^2}{c}\int_{-B}^{-B+c} \mathrm{Tr}\left\{\boldsymbol{T}(\omega)_i^{-1^T}\boldsymbol{T}(\omega)_i^{-1^*}\right\} d\omega. \tag{72}$$

This is the general equation for the noise level at the ith reconstructed channel.

We denote the sum of all $\overline{E\{|v_i^r(t)|^2\}}$ by σ_T^2. Using (71) we get

$$\begin{aligned} \sigma_T^2 &= \sum_{i=1}^{N}\overline{E\left\{|v_i^r(t)|^2\right\}} \\ &= \sum_{i=1}^{N}\frac{\sigma_v^2}{c}\int_{-B}^{-B+c}\sum_{k=1}^{m}\sum_{l=1}^{M}\left|T_{[m(i-1)+k],\,l}(\omega)^{-1}\right|^2 d\omega \\ &= \frac{\sigma_v^2}{c}\int_{-B}^{-B+c}\sum_{k=1}^{M}\sum_{l=1}^{M}\left|T_{kl}(\omega)^{-1}\right|^2 d\omega \\ &= \frac{\sigma_v^2}{c}\int_{-B}^{-B+c}\mathrm{Tr}\left\{\boldsymbol{T}(\omega)^{-1^T}\boldsymbol{T}(\omega)^{-1^*}\right\} d\omega. \end{aligned} \tag{73}$$

We now derive, using (72), a test that checks whether a VSE system is ill posed or well posed. This test is similar to the one suggested by Cheung and Marks [17] and Brown and Cabrera [18] for GSE systems. From (17), we see that the $(k,\, l)$th component of $\boldsymbol{T}(\omega)_i^{-1}$ is

$$T_{kl}(\omega)_i^{-1} = \frac{c}{2\pi} P_{i,\,l}(\omega+(k-1)c). \tag{74}$$

Substituting in (72) [actually in (71)], using a change of variables $\omega' = \omega+(k-1)c$ and noticing that the sum of the resulting integrals over the m intervals $[-B+(k-1)c,\, -B+kc]$ can be combined into an integral over the continuous interval $[-B,\, B]$, we get

$$\overline{E\left\{|v_i^r(t)|^2\right\}} = \frac{\sigma_v^2}{T}\frac{1}{2\pi}\int_{-B}^{B}\sum_{l=1}^{M}|P_{i,\,l}(\omega)|^2\, d\omega. \tag{75}$$

As noted above, a well posed VSE system is such that $E\{|v_i^r(t)|^2\}$ is bounded for every bounded σ_v^2. From (75), we conclude, similarly to [17] and [18], that a necessary condition for the well posedness of a VSE system is that all reconstruction filters $P_{i,\,l}(\omega)$ have a finite energy. It is also

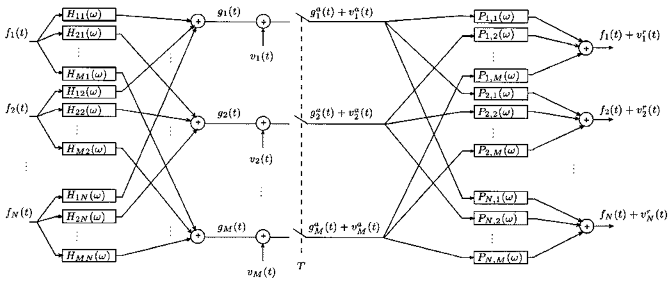

Fig. 7. Quantization noise in a VSE system.

a sufficient condition since we can easily see from (69) that $E\{|v_i^r(t)|^2\} \leq m\overline{E\{|v_i^r(t)|^2\}}$, i.e., it is finite when the $P_{i,l}(\omega)$ have a finite energy.

Using some simple matrix algebra, we get from (73)

$$\sigma_T^2 \leq \sigma_v^2 \frac{1}{c} \int_{-B}^{-B+c} \frac{M(MH_{\max})^{2(M-1)}}{|\det\{\boldsymbol{T}(\omega)\}|^2}\, d\omega \qquad (76)$$

where we assume that all $|H_{kl}(\omega)|$ are bounded from above by some finite number $H_{\max}$. Under this assumption, $|\det\{\boldsymbol{T}(\omega)\}| \geq \alpha > 0$ is a sufficient condition for well posedness. A similar condition was found by Brown and Cabrera [19] for GSE systems.

B. Optimal VSE Systems

We first wish to find the optimal VSE system in the sense of minimizing the total time-averaged mean square reconstruction error σ_T^2. Later on, we will also look for systems that minimize separately $\overline{E\{|v_i^r(t)|^2\}}$, which is the noise at a specific reconstructed signal. In order to get a meaningful answer, we need to impose power constraints on the filters of the VSE system since otherwise, the components of $\boldsymbol{T}(\omega)$, and therefore the output of the MIMO system, could be increased to any desired value, making the quantization noise insignificant. We will discuss several such constraints.

1) Minimal σ_T^2: Consider the following power constraint on the filters of the MIMO system:

$$M = \frac{1}{2B} \int_{-B}^{B} \sum_{k=1}^{M} \sum_{l=1}^{N} |H_{kl}(\omega)|^2\, d\omega. \qquad (77)$$

This power constraint was chosen because it is satisfied by the simple VSE system having N inputs and M outputs (integer M/N), in which $H_{kl}(\omega)$ is $e^{j\omega(k \bmod m)\pi/B}$ for $l = \lceil k/m \rceil$ and 0 otherwise ($m = M/N$). After sampling, this system is equivalent to Nyquist sampling of the N input signals $\{f_i(t)\}$.

The constraint (77) can also be expressed using the matrix $\boldsymbol{T}(\omega)$

$$Mm = \frac{1}{c} \int_{-B}^{-B+c} \mathrm{Tr}\{\boldsymbol{T}(\omega)^T \boldsymbol{T}(\omega)^*\}\, d\omega. \qquad (78)$$

We now minimize the right hand side of (73) under the power constraint (78). Let us start with a specific frequency ω and minimize $\mathrm{Tr}\{\boldsymbol{T}(\omega)^{-1^T} \boldsymbol{T}(\omega)^{-1^*}\}$, which is the integrand in (73). Suppose that

$$\mathrm{Tr}\{\boldsymbol{T}(\omega)^T \boldsymbol{T}(\omega)^*\} = Q(\omega)Mm. \qquad (79)$$

In addition, denote the M eigenvalues of the matrix $\boldsymbol{T}(\omega)^T \boldsymbol{T}(\omega)^*$ by $\lambda_n(\omega) n = 1, \cdots, M$. Recall that the trace of a matrix equals the sum of its eigenvalues, and therefore

$$\sum_{n=1}^{M} \lambda_n(\omega) = Q(\omega)Mm. \qquad (80)$$

Now, minimizing $\mathrm{Tr}\{\boldsymbol{T}(\omega)^{-1^T} \boldsymbol{T}(\omega)^{-1^*}\}$ is equivalent to minimizing $\sum_{n=1}^{M} (1/\lambda_n(\omega))$. The minimum, under the constraint that $\sum_{n=1}^{M} \lambda_n(\omega) = Q(\omega)Mm$, is achieved by

$$\lambda_n(\omega) = Q(\omega)m \qquad (81)$$

i.e., all eigenvalues of $\boldsymbol{T}(\omega)^T \boldsymbol{T}(\omega)^*$ should be equal. The minimal value of $\sum_{n=1}^{M} (1/\lambda_n(\omega))$ is $N/Q(\omega)$.

Returning to (73)

$$\begin{aligned} \sigma_T^2 &= \sigma_v^2 \frac{1}{c} \int_{-B}^{-B+c} \mathrm{Tr}\left\{\boldsymbol{T}(\omega)^{-1^T} \boldsymbol{T}(\omega)^{-1^*}\right\} d\omega \\ &= \sigma_v^2 \frac{1}{c} \int_{-B}^{-B+c} \sum_{n=1}^{M} \frac{1}{\lambda_n(\omega)}\, d\omega \\ &\geq \sigma_v^2 \frac{1}{c} \int_{-B}^{-B+c} \frac{N}{Q(\omega)}\, d\omega. \end{aligned} \qquad (82)$$

Minimizing the right-hand side under the power constraint $Mm = (1/c)\int_{-B}^{-B+c} Q(\omega)Mm\, d\omega$ of (78) leads to

$$Q(\omega) = 1 \qquad \omega \in [-B, -B + c]. \qquad (83)$$

Thus optimal performance is attained by

$$\lambda_n(\omega) = m \qquad \omega \in [-B, -B + c]. \qquad (84)$$

The only matrix in which all eigenvalues λ_n equal to m is $m\boldsymbol{I}$. Therefore, $\boldsymbol{T}(\omega)$ of an optimal VSE system must be $\sqrt{m}$ times a unitary matrix for every $\omega \in [-B, -B + c]$. Such a system produces a total noise of $N\sigma_v^2$. Thus, under the power conditions of (77) or (78), we always have

$$\sigma_T^2 \geq N\sigma_v^2 \qquad (85)$$

where equality occurs only if (84) is satisfied. The optimal value of the reconstruction noise *does not* depend on m.

We now define the noise amplification factor A_{ϵ_i}

$$A_{\epsilon_i} = \frac{\overline{E\{|v_i^r(t)|^2\}}}{\sigma_v^2}. \qquad (86)$$

An infinite A_{ϵ_i} suggests that the system is ill posed or even that the matrix $\boldsymbol{T}(\omega)$ is singular. From (85), we see that

$$\sum_{i=1}^{N} A_{\epsilon_i} \geq N. \qquad (87)$$

This means that when we demand equal reconstruction noise, or equal noise amplification, for all N reconstructed signals, we get $A_{\epsilon_i} \geq 1$.

We can also determine the minimal possible value of a specific A_{ϵ_i} by considering the ith reconstructed signal as coming from a GSE system with one input and M outputs sampled N times faster than necessary. Using the result above, and the fact that the noise is averaged over the N possible versions of the reconstructed signal, we get

$$A_{\epsilon_i} \geq \frac{1}{N}. \tag{88}$$

2) Power Constraint on the Inputs: Consider the following more restrictive power constraint. For every $l = 1, \cdots, N$

$$m = \frac{M}{N} = \frac{1}{2B} \int_{-B}^{B} \sum_{k=1}^{M} |H_{kl}(\omega)|^2 \, d\omega. \tag{89}$$

This constraint is called power constraint on the inputs since it involves all of the M filters $H_{kl}(\omega)$ receiving $f_l(t)$ at their input. Note that this constraint does not contradict the previous power constraint of (77).

Under this power constraint, we can still attain $\sigma_T^2 = N \cdot \sigma_v^2$, i.e., $\sum_{i=1}^{N} A_{\epsilon_i} = N$. However, now, the minimal value of A_{ϵ_i} is 1 for all i. The solution that simultaneously attains $A_{\epsilon_i} = 1$ also attains the minimal σ_T^2. This is shown by considering, again, the ith reconstructed signal as coming from a GSE system with one input and M outputs, sampled N times faster than necessary, but now, the power constraint on the inputs forces the power to be N times smaller, and therefore, here, $A_{\epsilon_i} \geq 1$.

3) Power Constraint on the Outputs: Another more restrictive power constraint is "power constraint on the outputs." For every $k = 1, \cdots, M$

$$1 = \frac{1}{2B} \int_{-B}^{B} \sum_{l=1}^{N} |H_{kl}(\omega)|^2 \, d\omega \tag{90}$$

i.e., the same power in all M outputs.

We show that this constraint allows the attainment of the minimal value $A_{\epsilon_i} = 1/N$, but the better noise amplification factor of one signal comes at the expense of a worse amplification for the other signals. We start our analysis with filters $H_{kl}(\omega)$ satisfying

$$\frac{1}{N} = \frac{1}{2B} \int_{-B}^{B} |H_{kl}(\omega)|^2 \, d\omega \tag{91}$$

for every k and l. We define N positive amplification factors $\sqrt{a_l}$ and demand that

$$N = \sum_{l=1}^{N} a_l. \tag{92}$$

Suppose we have a system satisfying (91) and having all $A_{\epsilon_i} = 1$. Such a system can always be found, e.g., by using ideal subband filters. By amplifying the ith input signal by a factor $\sqrt{a_i}$, we get $A_{\epsilon_i} = 1/a_i$, and therefore, we can control the noise amplification factor of each signal.

When the amplifying factors a_l are used, the total noise is

$$\sigma_T^2 = \sigma_v^2 \sum_{l=1}^{N} \frac{1}{a_l}. \tag{93}$$

Using the Lagrange multipliers method, it is easy to see that the minimum value of σ_T^2, under the constraint of (91), is achieved only when all A_{ϵ_i} are 1. This means that improving the noise of a certain output leads to a higher loss in the other outputs.

The situation discussed above can be demonstrated by the following example. Let $N = 10$, and suppose that initially, $A_{\epsilon_l} = 1$ for all values of l. We want to have $A_{\epsilon_1} = 0.5$, so we set a_1 to be 2. Because $10 = \sum_{l=1}^{10} a_l$, we have $8 = \sum_{l=2}^{10} a_l$. If we choose $a_l = 8/9$ for $l = 2, \cdots, 10$, we find that

$$\sigma_T^2 = \sigma_v^2 \sum_{l=1}^{10} \frac{1}{a_l} = \sigma_v^2 \left(\frac{1}{2} + 9 \cdot \frac{9}{8} \right) = 10.63\sigma_v^2 > N\sigma_v^2.$$

Note that when (91) is satisfied, we have such control on the reconstruction noise even if the VSE system is not optimal. This enables control of the reconstruction quality of some of the signals at the expense of the rest of the signals while keeping the output level equal (which simplifies quantization).

We end this section by pointing out a simple example of optimal VSE system, with $N = M = 2$. Let $\boldsymbol{T}(\omega)$ be

$$\boldsymbol{T}(\omega) = \begin{bmatrix} \cos\,\alpha(\omega) & -\sin\,\alpha(\omega) \\ \sin\,\alpha(\omega) & \cos\,\alpha(\omega) \end{bmatrix}. \tag{94}$$

Since $\boldsymbol{T}(\omega)$ is unitary, we have $\lambda_1 = \lambda_2 = 1$ for every $\omega \in [-B, -B + c]$. Interesting cases can be obtained by choosing various values for $\alpha(\omega)$. For example, using $\alpha(\omega) = 0$ for $\omega \in [-B/2, B/2]$, and $\pi/2$ elsewhere, leads to an optimal VSE system composed of ideal bandpass filters.

VI. Conclusion

In this paper, we have shown that under certain conditions on the sampling rates, it is possible to reconstruct N bandlimited signals, which are the inputs to a MIMO LTI system, from periodic samples of the M outputs of the system ($M \geq N$) while keeping the total rate to be N times the Nyquist rate. We discussed equal uniform sampling, nonequal uniform sampling, and equal periodic nonuniform sampling. Interestingly, for equal uniform sampling, reconstruction is possible only if the expansion is by an integer factor (M/N is an integer). It is also shown that for any $M \geq N$, there is at least one sampling combination in which the sampling periods are multiples of the Nyquist period that allow reconstruction. In all cases, we also derived the explicit interpolation formulas.

The paper contains a noise sensitivity analysis, which determine the necessary and sufficient conditions for minimum time averaged mean square reconstruction error. We defined the noise amplification factor, which provides a quantitative comparison of VSE systems under a power constraint.

Appendix
Hall's Marriage Problem, Perfect Matching of Bipartite Graph, and the Invertibility of $\boldsymbol{T}(\omega)$

As we show below, determining whether a unique reconstruction is possible, i.e., whether the matrix $\boldsymbol{T}(\omega)$ can be made invertible for all $\omega \in [-B, -B + c]$, is equivalent to Hall's *marriage theorem* [12], [13]. This theorem is also known as the

SDR theorem of Hall, where SDR stands for "systems of distinct representatives" [14], [20], [21]. The theorem deals with the necessary and sufficient condition for selecting a distinct set from the set of members $\boldsymbol{V} = \{V_1, V_2, \cdots, V_m\}$ such that there is a one-to-one correspondence between each of the components of the chosen set and the components of a given set $\boldsymbol{S} = \{S_1, S_2, \cdots, S_n\}$, which is a set of subsets of $\boldsymbol{V}$.

For example let $\boldsymbol{V} = \{1, 2, 3, 4, 5, 6\}$, $\boldsymbol{S} = \{S_1, S_2, S_3, S_4, S_5\}$, where $S_1 = \{2, 4, 6\}$, $S_2 = \{3, 6\}$, $S_3 = \{3, 5\}$, $S_4 = \{1, 5\}$, and $S_5 = \{1, 2\}$. The set $\{2, 6, 3, 5, 1\}$ is a set of distinct representatives, where

- 2 represents S_1;
- 6 represents S_2;
- 3 represents S_3;
- 5 represents S_4;
- 1 represents S_6.

In case we had the sets $S_1 = \{1, 2, 3\}$, $S_2 = \{2, 3\}$, $S_3 = \{2, 4\}$, $S_4 = \{1, 4\}$, and $S_5 = \{1, 2\}$, we could not find a set of distinct representative since only four elements of $\boldsymbol{V}$ participate in $\boldsymbol{S}$, whereas there are five sets in $\boldsymbol{S}$.

Hall's SDR theorem states that an SDR exists for $\boldsymbol{S} = \{S_1, S_2, \cdots, S_n\}$ if and only if every collection of k sets of $\boldsymbol{S}$ contains at least k distinct members for every possible value of k, i.e., for all $k = 1, \cdots, n$ (see [20]).

In other words, the union of every combination of k sets must contain at least k elements. This is a necessary condition since otherwise, we would not be able to assign a distinct element to every set as shown in above example. The proof that this is a sufficient condition can be found, for example, in [21].

As noted in [22], this is similar to condition i) of Section III, which demands that for any possible choice of k equations, k must be less than or equal to the number of all the unknowns appearing in the k equations. The equations are the sets S_i, and the unknowns are the components of the set $\boldsymbol{V}$.

The SDR problem is equivalent to the problem of *matching* in a bipartite graph [14]. A bipartite graph is a graph composed of two disjoint subsets of vertices such that no vertex in a subset is adjacent to vertices in the same subset [23], i.e., it can be considered as two columns of vertices. We can consider one of them (the left one for example) as representing the sets $\boldsymbol{S}$ and the other (the right one) as representing the set $\boldsymbol{V}$. When discussing testing of a sampling combination, the first represents the rows of the matrix $\boldsymbol{T}(\omega)$ (i.e., the equations), and the other represents the columns (i.e., the unknowns). An edge of the graph connects the ith left vertex to the jth right vertex if V_j is a member in S_i. In our case, when we are testing a sampling combination, an edge connecting the ith left vertex to the jth right vertex exists if $T_{ij}(\omega)$ is nonzero, i.e., if the jth unknown appears in the ith equation. A perfect matching of bipartite graph having the same number of vertices in both sides is a matching in which all vertices of the two sides of the graph are connected with only one edge. This is similar to finding a distinct representative for each of the S_i sets of the left, in the list of the V_i's on the right, when both sets have the same number of elements. Here, we clearly see the equivalence of the SDR problem and the perfect matching problem. Such a perfect matching can be described as a permutation σ, where $\sigma(i) = j$ represents the edge connecting S_i and V_j. We denote the permutation matrix by A_σ. The (i, j)th component of A_σ is one only if $\sigma(i) = j$ and zero otherwise.

We now show that a necessary and sufficient condition, allowing an invertible matrix $\boldsymbol{T}(\omega)$, is the existence of a perfect matching in the corresponding bipartite graph.

It is a necessary condition since the determinant of any matrix $\boldsymbol{T}$ of size $mN \times mN$ can be written as $\det\{\boldsymbol{T}\} = \sum_{\text{all } (mN)! \text{ possible } \sigma} \text{sign}(\sigma) \prod_{i=1}^{mN} T_{i, \sigma(i)}$ (see [14] and [24]). Thus, if no permutation exists in which all components $T_{i, \sigma(i)}$ are nonzero, the determinant must be zero.

Sufficiency is easily shown since having a perfect matching means that at least one permutation matrix A_σ (the one that describes the perfect matching) exists, where all mN components $T_{i, \sigma(i)}$ are nonzero. If we choose these components to be 1 and the rest of the components in $\boldsymbol{T}(\omega)$ to be zero, we have $\boldsymbol{T}(\omega) = A_\sigma$, which is definitely invertible.

Hall's conditions can be verified using the "Hungarian method" of König and Egerváry, which requires $O((mN)^3)$ operations or even by $O((mN)^{5/2})$ operations of Hopcroft and Karp (see [14]) or Even [15]. This approach has the advantage that the question whether or not unique reconstruction is possible can be answered without choosing specific values for the MIMO system $\boldsymbol{H}(\omega)$.

ACKNOWLEDGMENT

The authors would like to thank N. Alon for pointing out the relation between the unique reconstruction problem and the problem of perfect matching in bipartite graphs.

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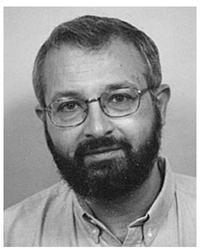

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